Computational Issues in Nonlinear Control and Estimation

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They dealt with control and estimation problems in

State Space Form.

Linear State Space Control Theory

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And there was much lamenting the gap between the linear state space theory and the linear frequency domain practice.

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In the later 1970s LINPAK and a related package called EISPAK were replaced and supplanted by LAPACK which also uses BLAS.

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MATLAB was first adopted by researchers and practitioners in control engineering, Little's specialty, but quickly spread to many other domains.

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So special purpose software is becoming available for some nonlinear systems.

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- **General Purpose**

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It should be noted that the numerical optimization community has already developed software with these properties for a wide variety of applications like Mixed Intger Linear Programs (i.e., GUROBI and CPLEX) and Nonlinear Programs (i.e., KNITRO and IPOPT). For optimzation solver benchmarks see http://plato.la.asu.edu/bench.html

Smooth Nonlinear Systems

A smooth nonlinear system is of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

with state $x \in \mathbb{R}^{n\times 1}$, input $u \in \mathbb{R}^{m\times 1}$ and output $y \in \mathbb{R}^{p\times 1}$

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There may be state and control constraints of the form

$$g(x,u) \leq 0$$

Any problem with other than linear equality constraints is inherently nonlinear even if $f(x,u), h(x,u)$ **are linear in** x, u **.**

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- **Finding a feedforward and feedback that tracks a reference trajectory.**
- **Your favorite problem.**

Linear Solutions

These are fundamental problems for both linear and nonlinear systems. For linear systems we have theoretical solutions that are easily implemented numerically. Here is a few.

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It can be difficult to solve a problem that has many solutions. Introducing an optimality criterion narrows the search.

Optimal Stabilization

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A classic way to find a feedback $u = \kappa(x)$ that stabilizes a nonlinear system to an operating point is to pose and solve an infinite horizon optimal control problem. Choose a Lagrangian $l(x, u)$ that is nonnegative definite in x, u and positve definite in u and

$$\min_{u(\cdot)} \int_0^\infty l(x, u) \; dt$$

subject to

$$\begin{aligned} \dot{x} &= f(x, u) \\ x(0) &= x^0 \end{aligned}$$

Optimal Stabilization

The reason is the optimal solution is given by a feedback $u(t) = \kappa(x(t))$ and the optimal cost $\pi(x^0) \geq 0$ starting at x^0 is a potential Lyapunov function for the closed loop system

$$\frac{d}{dt}\pi(x(t)) \;\; = \;\; -l(x(t), \kappa(x(t))) \leq 0$$

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If $x \neq 0$ and $\pi(x) \leq c$

$$\begin{aligned} 0 &< \pi(x) \\ 0 &> \frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) \end{aligned}$$

HJB Equations

If there exist smooth solutions $\pi(x), \kappa(x)$ to the Hamilton-Jacobi-Bellman equations

$$
\begin{aligned}
0 &= \min_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\} \\
\kappa(x) &= \operatorname{argmin}_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\}
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then these are the optimal cost and optimal feedback.

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The control Hamiltonian is

$$\mathcal{H}(\lambda, x, u) = \lambda' f(x,u) + l(x,u)$$

HJB Equations

If the control Hamiltonian is strictly convex in u for every λ, x then the HJB equations simplify to

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If $R(x) > 0$ and

$$\begin{aligned} f(x, u) &= f_0(x) + f_1(x) u \\ l(x, u) &= q(x) + \frac{1}{2} u' R(x) u \end{aligned}$$

then $\mathcal{H}(\lambda, x, u)$ is strictly convex in u for every λ, x.

Linear Quadratic Regulator

About the only time the HJB equations can be solved in closed form is the so-called Linear Quadratic Regulator (LQR)

$$\min_{u(\cdot)} \frac{1}{2} \int_0^\infty x'Qx + 2x'Su + u'Ru \; dt$$

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The optimal cost is $\pi(x) = \frac{1}{2}x'Px$ and optimal feedback is $u = Kx$ where P, K satisfy

$$\begin{aligned} 0 &= F'P + PF + Q - (PG+S)R^{-1}(PG+S)' \\ K &= -R^{-1}(PG+S)' \end{aligned}$$

Jacobian Linearization

Standard engineering practice is to approximate the nonlinear dynamics

$$\dot{x} \;=\; f(x,u)$$

by its Jacobian linearization

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Then $\frac{1}{2}x'Px$ and $u = Kx$ are approximations to the true optimal cost $\pi(x)$ and optimal feedback $u = \kappa(x)$.

Lyapunov Argument

If

$$f(x,u) = Fx + Gu + O(x,u)^2$$

then

$$\begin{aligned}\frac{d}{dt}\left(x'(t)Px(t)\right) &= -x'(t)\left(Q + (PG+S)R^{-1}(PG+S)'\right)x(t) \\ &\quad + O(x(t))^3\end{aligned}$$

so on some neighborhood of $x = 0$ the feedback $u = Kx$ stabilizes the nonlinear system.

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Kicker: How big is the neighborhood?

Al'brekht's Method

In 1961 Ernst Al'brekht noticed that first HJB equation

$$0 \;=\; \frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) + l(x, \kappa(x))$$

is singular at the operating point $x = 0, u = 0$ because $f(0, 0) = 0$.

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This means that it is amenable to power series methods. Let $\pi^{[k]}(x)$ denote a homogeneous polynomial of degree k and consider the linear operator

$$\pi^{[k]}(x) \mapsto \frac{\partial \pi^{[k]}}{\partial x}(x)(F + GK)x$$

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$$\pi^{[k]}(x) \;\mapsto\; \frac{\partial \pi^{[k]}}{\partial x}(x)(F + GK)x$$

The eigenvalues of this operator are the sums of k eigenvalues of $F + GK$ so if the latter are all in the open left half plane than so are the former.

Al'brekht's Method

Al'brekht plugged the Taylor polynomial expansions

$$
\begin{aligned}
f(x,u) &\approx Fx + Gu + f^{[2]}(x,u) + \ldots + f^{[d]}(x,u) \\
l(x,u) &\approx \frac{1}{2}\left(x'Qx + 2x'Su + u'Ru\right) + \ldots + l^{[d+1]}(x,u) \\
\pi(x) &\approx \frac{1}{2}x'Px + \pi^{[3]}(x) + \ldots + \pi^{[d+1]}(x) \\
\kappa(x) &\approx Kx + \kappa^{[2]}(x) + \ldots + \kappa^{[d]}(x,u)
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into the HJB equations and collected terms of like degrees.

Al'brekht's Method

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Al'brekht's Method

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There are similar linear equations for the higher degree terms, $\pi^{[k+1]}(x)$ and $\kappa^{[k]}(x)$.

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Here is an example.

Optimal Stabilization of a Rigid Body

The example was fairly simple with $n = 4$ states and $m = 2$.

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Our goal is to optimally stabilize the body to a desired constant velocity and desired attitude.

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hjb_time_3 = 4.0260 seconds

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This is 26.7466 minutes. The number of monomials of degrees one through five in $n + m = 18$ variables is 33648.

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- **What if there are discontinuities?**

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A variation on Model Predictive Control (MPC) is an answer to the first question.

In MPC we don't try to solve off-line for the optimal cost $\pi(x)$ and optimal feedback $\kappa(x)$ to an infinite horizon optimal control problem.

Instead we solve on-line a discrete time finite horizon optimal control problem for our current state. Suppose $t = t_0$ and $x(t_0) = x^0$, we choose a horizon length T and a Lagrangian $L(x,u)$ and seek to find the optimal control sequence $\mathbf{u}^*_{t_0} = (u^*(t_0), \dots, u^*(t_0 + T - 1))$ that minimizes

$$\sum_{t=t_0}^{t_0+T-1} L(x(t), u(t)) + \Pi(x(t_0 + T))$$

$$x(t+1) = F(x(t), u(t))$$

Model Predictive Control

There is a discrete time version of Al'brekht for the corresonding infinite horizon optimal control problem implemented in `dpe.m` **that yields Taylor polynomials for the infinite time optimal cost $\mathbf{\Pi}(x)$ and optimal feedback $\mathcal{K}(x)$.**

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We use $\Pi(x)$ as our terminal cost for the discrete time finite horizon optimal control problem for then the finite horizon and infinite horizon costs are the same.

The discrete time finite horizon optimal control problem is a nonlinear program which we pass to a fast solver like KNITRO or IPOPT. The solver returns $\mathbf{u}^*_{t_0}$, we implement the feedback $u(t_0) = u^*(t_0)$ and move one time step. We increment t_0 by one which slides the horizon forward one time step and repeat the process.

Adaptive Horizon Model Predictive Control

But how do we know that the horizon T is long enough so that the endpoint $x(t_0 + T)$ is in the domain where $\Pi(x)$ is a valid Lyapunov function for closed loop dynamics under the feedback $u = \mathcal{K}(x)$?

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If the are not satisfied we increase the horizon.

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If there are constraints $G(x, u) \leq 0$ we can take the same approach as long as they are not active at the operating point $G(0, 0) < 0$.

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And we check that the constraints and the Lyapunov conditions are satisfied on the computed extension.

Examples

We seek to stabilize a double pendulum to the upright position using torques at each of the pivots.

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The damping coefficients at the two joints are both 0.5 **s**$^{-1}$ **.**

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The continuous time dynamics is discretized using Euler's method with time step 0.1 s. assuming the control is constant throughout the time step.

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We do not move one time step forward if the feasibility and Lyapunov conditions do not hold over the extended state trajectory but instead we increased N by 1 and recomputed from the same x.

Example 1

$$x^0 \;=\; (0.5\pi, -0.5\pi, 0, 0)'$$

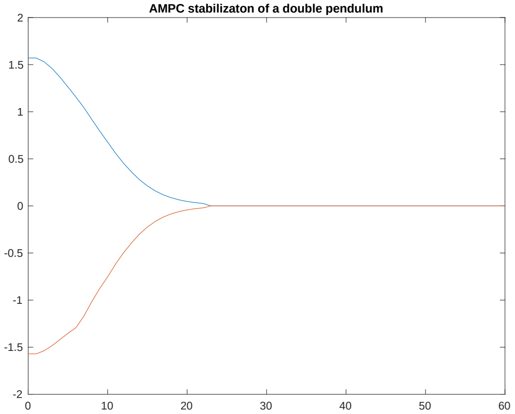

Figure: Angles Converging to the Vertical

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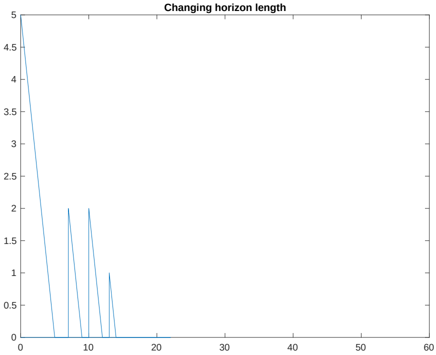

Figure: Adaptively Changing Horizon

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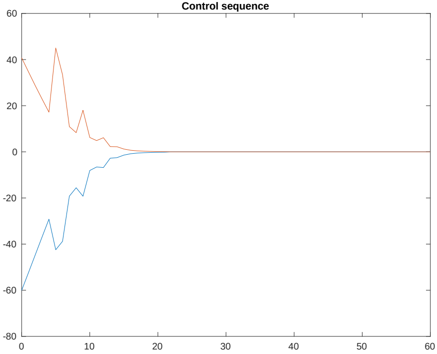
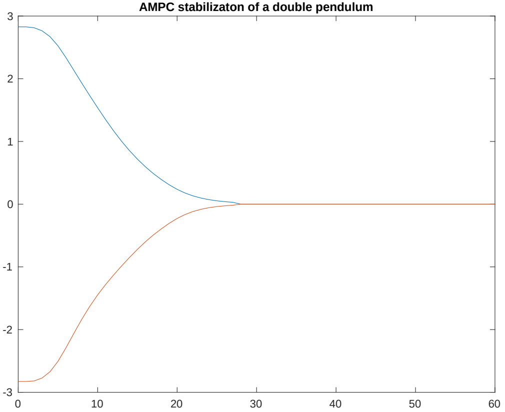


Figure: Control sequence

Example 2

$$x^0 \;=\; (0.9\pi, -0.9\pi, 0, 0)'$$



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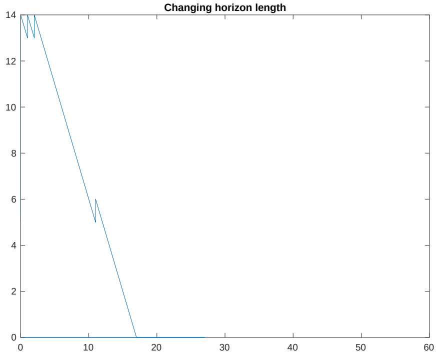

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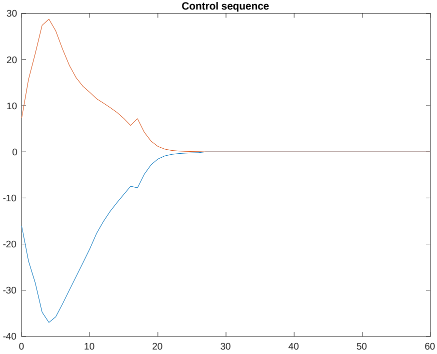

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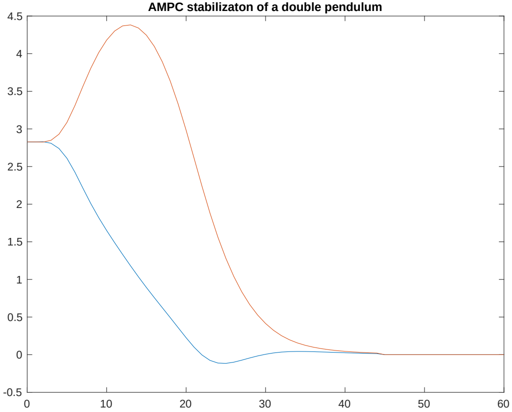

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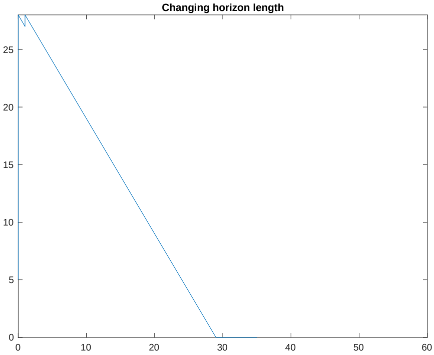

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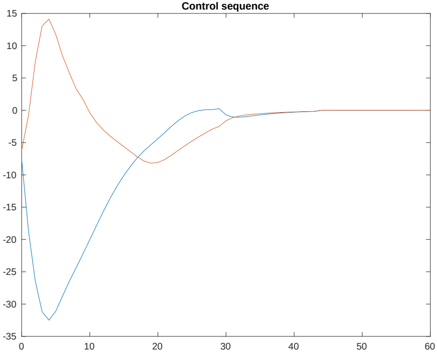

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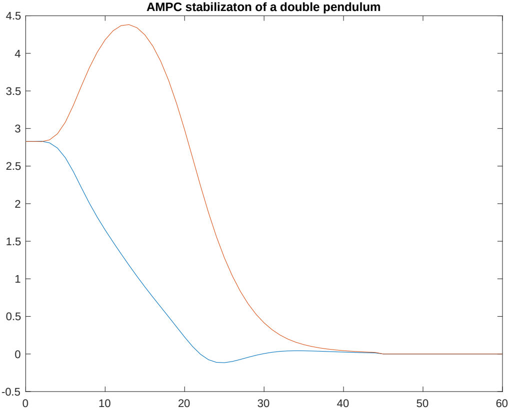
Example 4

$$
\begin{aligned}
x^0 &= (0.9\pi, 0.9\pi, 0, 0)' \\
|u_1| &\leq 10 \\
|u_2| &\leq 10
\end{aligned}
$$



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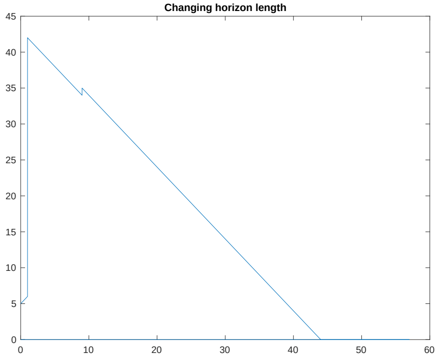

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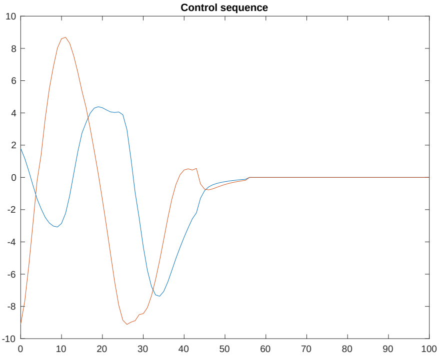

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Our Slogan

Think Mathematically

Our Slogan

Think Mathematically

Act Computationally

Conclusion

Thank You